Restricted-Weight Minimum-Dilation Spanners on Three Points

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Abstract

Given a planar point set P and a parameter $L > 0$, we are interested in finding a Euclidean graph G, possibly using Steiner vertices, that has total weight at most L and minimizes the maximum dilation between any pair of points $p, q \in P$. The dilation between two points is the ratio between their graph distance and their Euclidean distance.

While this problem can be approached using convex optimization, the geometry of solutions is not yet well understood. We investigate this problem for the case $P = \{A, B, C\}$ of three points. In this case the solution consists of a triangle $\Delta A'B'C'$ and edges AA', BB' and CC'. We show that if A, B and C are the vertices of an equilateral triangle ΔABC, then $\Delta A'B'C'$ is equilateral and centered in ΔABC with its vertices on the bisectors of $\angle ABC, \angle ACB, \angle BAC$. We further analyze the solution for the case that ΔABC is isosceles and L is small.

1 Introduction

A Euclidean graph is a graph in which each vertex is a point in $\mathbb{R}^n$, and each edge (p, q) has weight equal to the Euclidean distance $|pq|$ between p and q. Let $G(V, E)$ be a Euclidean graph and let $d_G(p, q)$ denote the shortest distance between $p, q \in V$ in graph G, that is, the smallest total weight of any path from p to q. Let furthermore $D_G(p, q) = \frac{d_G(p,q)}{|pq|}$ be the dilation between $p, q \in V$ over graph G. A *spanner* of a set of points $P \in \mathbb{R}^n$ is then defined as a connected graph $G = (V, E)$ such that $P \subset V$. Depending on the context, one may require $V = P$, or one may allow V to contain *Steiner points*, that is, points that are not in P. The dilation D_G of the spanner is its maximum dilation over all pairs of points from P, that is, $D_G = \max_{p,q \in P} D_G(p, q)$. A *$t$-spanner* is a spanner G with dilation $D_G \leq t$. Spanners and related algorithms have extensively been described by Narasimhan and Smid [15].

Spanners form an important concept in computational geometry. Geometric spanners have various applications in for example the searching of metric spaces [16], the distribution of messages in networks [10], and the creation of approximate distance oracles [11]. Generally, geometric spanners can be utilized to approximate more complex networks, allowing for more time-efficient approximation algorithms. In the theoretical analysis of such algorithms, dilation is often used to prove bounds on the accuracy of said algorithms.

Spanners are frequently studied under further constraints regarding a variety of measures. Commonly considered measures include the number of edges in the spanner, diameter of the spanner and the total weight of the spanner. The basic spanner problem concerns minimizing the number of edges for a given desired dilation [15]. The greedy spanner, as introduced by Althöfer et al. [2], has been proven asymptotically optimal for the basic spanner problem [14]. Further approximation algorithms aim for asymptotic complexities of $O(n)$ and $O(|MST|)$ for the number of edges and the weight respectively [1, 3, 4, 5, 15]. An Integer Linear Program formulation for the basic spanner problem, was given by Sigurd and Zachariasen [17]. These results are for the setting without Steiner vertices.

The knowledge on other settings is less extensive. In this report, we investigate spanners with Steiner vertices and constrained in weight and with minimal dilation. The related Minimum-Weight Spanner Problem (without Steiner points), considering the minimal weight for a given dilation, has been proven to be NP-hard [7]. Generally, little is known about the exact structure of optimal spanners. While minimizing the total weight of the spanner, considering Steiner points naturally allows for better results. Limited work is available on settings allowing for Steiner points [6].

We study the optimal geometric structure of solutions in a weight-restricted setting analytically. Specifically, we consider the *Restricted-Weight Minimum-Dilation Spanner Problem with Steiner Points* on geometric problem instances consisting of three points in $\mathbb{R}^2$. Given a maximum weight L, where weight is defined as the sum over the lengths of all edges, we thus want to find the spanner over $P \cup S$, for S a set of Steiner points that is optimal, that is the spanner minimizing the maximum dilation between any two points in P. Previous work on this problem by Kooijmans [13] and Verstege [18] has focused on providing algorithms for determining optimal spanners based on convex optimization.

We characterize the general topological structure of optimal spanners for all problem instances with $|P| = 3$.

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For the case that the points in P are the vertices of an equilateral triangle, we provide a complete geometric description of the optimal spanner. Figure 1 contains examples of optimal spanners for this case. Likewise, we provide a description for the case of an isosceles triangle and low maximum weight L.

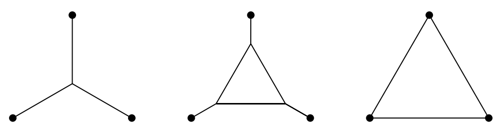

Figure 1: Optimal spanners for P consisting of the vertices of an equilateral triangle for increasing maximum weight

Analyzing geometric properties of optimal spanners serves the development of fast and well-performing approximation algorithms, by providing exploitable substructures and generally furthering the understanding of the problems at hand. As such, this work aims to aid further research into weight-restricted geometric spanners with Steiner vertices.

2 General Triangle

Let $P \subset \mathbb{R}^2, |P| = 3$, such that the set forms a triangle. Let these points be labeled A, B, C. As $G = (P \cup S, E)$ must be a connected graph, there must exist paths between each pair of points, which potentially overlap. Let $\alpha, \beta, \gamma : [0, 1] \to \mathbb{R}^2$ be curves describing the paths between A and B, B and C, and C and A respectively. Let furthermore $\alpha[x, y]$ denote the subcurve from $\alpha(x)$ to $\alpha(y)$ for $0 \leq x \leq y \leq 1$.

The proofs of Lemmas 1 to 5 are given in the appendix.

Lemma 1 *An optimal spanner has a maximal dilation of 1, or the total cost is maximal, thus* $\sum_{e \in E} |e| = L$.

Lemma 2 *In an optimal spanner, curves* α, β, γ *are injective.*

As a result of Lemma 2, we can denote $\alpha[x, y]$ by $\alpha[X \to Y]$ for $X = \alpha(x), Y = \alpha(y)$ with $0 \leq x, y \leq 1$. We furthermore denote the subcurve $\alpha[x, y]$ in reverse direction by $\alpha[Y \to X]$. We similarly denote the subcurves $\alpha[X \to Y]$ and $\alpha[Y \to X]$ excluding the endpoints by $\alpha(X \to Y)$ and $\alpha(Y \to X)$ respectively.

Lemma 3 *Let* X, Y *be points shared by curves* α, β *in an optimal spanner. Then*

$$\alpha[X \to Y] = \beta[X \to Y]$$

Similarly for pairs α, γ *and* β, γ.

Thus, between two points shared by two curves, the two curves consist of identical point sets.

Lemma 4 *In an optimal spanner for* $L \leq |AB| + |AC| + |BC|$, *curves* α, β, γ *consist of straight line segments between* A, B, C *and additional Steiner points.*

Lemma 5 *In an optimal spanner,* α, β, γ *are contained in the convex hull of* A, B, C.

Lemma 6 *An optimal spanner has the following form: a triangle with each vertex connected to exactly one of the points* A, B, C.

Proof. Let curve α be given arbitrarily. We then consider curves β, γ. As $\alpha(1) = \beta(0) = B$, and by Lemma 3, there must exist a point B' such that $\alpha[B \to B'] = \beta[B \to B'] = \alpha \cap \beta$. Analogously, there exist a point C' such that $\beta[C \to C'] = \gamma[C \to C'] = \beta \cap \gamma$ and a point A' such that $\alpha[A \to A'] = \gamma[A \to A'] = \alpha \cap \gamma$.

If $\alpha(A' \to B') \cap \beta \neq \emptyset$, let $X \in \alpha(A' \to B') \cap \beta$. By Lemma 2, $\alpha(A' \to B') \cap \beta(B \to B') = \emptyset$. Then by Lemma 3, $\alpha[X \to B']$ and $\beta[X \to B']$ must coincide. However, as in this case $\alpha[X \to B] = \beta[X \to B]$ and $B' \in \alpha(X \to B)$, this contradicts with the definition of B'. Therefore, $\alpha(A' \to B') \cap \beta = \emptyset$. Similarly, $\alpha(A' \to B') \cap \gamma = \emptyset$.

Thus in any optimal spanner, $\alpha(A' \to B') \cap \beta = \alpha(A' \to B') \cap \gamma = \emptyset$. Analogously, $\beta(B' \to C') \cap \alpha = \beta(B' \to C') \cap \gamma = \emptyset$ and $\gamma(A' \to C') \cap \alpha = \gamma(A' \to C') \cap \beta = \emptyset$.

We then, using Lemma 5, conclude that any optimal spanner must have the topology as shown in Figure 2. From Lemma 4 it follows that the curves between A and A', B and B', C and C', A' and B', B' and C', A' and C', must be line segments. Thus the final form is as shown in Figure 3. $\square$

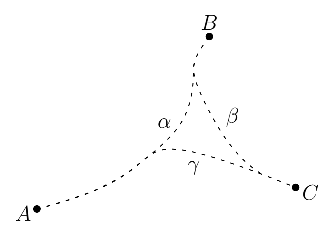

Figure 2: Optimal topology as proven in Lemma 6

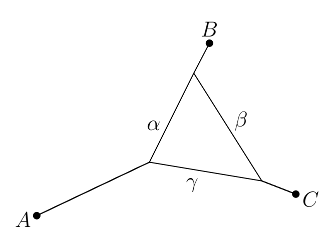

Figure 3: Optimal form as proven in Lemma 6

As used in the proof of Lemma 6, let A' be the point such that $\alpha[A \to A'] = \gamma[A \to A']$ and $|\alpha[A \to A']|$ maximal. Similarly, let B' be the point such that $\alpha[B \to B'] = \beta[B \to B']$ and $|\beta[B \to B']|$ maximal, and let C' be the point such that $\beta[C \to C'] = \gamma[C \to C']$ and $|\gamma[C \to C']|$ maximal.

Lemma 7 *In an optimal spanner, A', B', C' are distinct or $A' = B' = C'$.*

Proof. Assume that there exists an optimal spanner where this is not the case. Without loss of generality, assume that $A' = B'$, and thus $C' \neq A' = B'$. Then, $\beta[C, A'] = \gamma[C, A']$. Furthermore, by Lemma 6, and as A', C' are distinct, $|\gamma[C, A']| > |\gamma[C, C']|$. But this contradicts with the definition of C'. Thus, our assumption cannot hold. □

Lemma 8 *In an optimal spanner, $A' = B' = C'$ or $D_G(A, B) = D_G(A, C) = D_G(B, C)$.*

Proof. Assume $D_G(A, B) > D_G(B, C)$ in an optimal spanner. Then segment $B'C'$ can be moved parallelly inwards towards the center of triangle $A'B'C'$ to reduce total cost, while not altering the maximal dilation. Then, from Lemma 1, it follows that the spanner cannot be optimal. Thus, $D_G(A, B) \leq D_G(B, C)$ must hold for an optimal spanner with A', B', C' distinct. Similarly, because of the ability to move segment $A'B'$ parallelly inwards, $D_G(A, B) \geq D_G(B, C)$. By analogous arguments, for a spanner to be optimal with A', B', C' distinct, $D_G(A, B) = D_G(A, C) = D_G(B, C)$ must hold. □

The *dilation center* X^* of three points A, B, C is defined as the point X that minimizes the dilation of the graph with edges AX, BX and CX, where $P = A, B, C$. We call G^*, the graph with edges AX^*, BX^* and CX^*, the *minimum-dilation star*. The point X^* has been entered into the Encyclopedia of Triangle Centers as X(3513) [12]. It has been named the 1st Dilation Center, and was contributed by Eppstein [8, 9], who also observed that $D_{G^*}(A, B) = D_{G^*}(B, C) = D_{G^*}(A, C)$. Let L^* be the weight of G^*, that is, $L^* = |AX^*| + |BX^*| + |CX^*|$.

Lemma 9 *For any point M in the interior or on the boundary of ΔABC, $|AM| + |BM| + |CM| < |AB| + |AC| + |BC|$.*

See appendix for the proof of Lemma 9.

Lemma 10 *Let L_{MST} be the weight of the minimum Steiner tree over P. Then, for any weight L such that $L_{MST} \leq L \leq L^*$, any optimal spanner satisfies $A' = B' = C'$.*

Proof. We first show that for $L = L^*$, G^* is optimal. Assume that there exists another spanner G with cost at most L^*, and smaller maximum dilation. By its definition, G^* is optimal among spanners satisfying $A' = B' = C'$. Thus, in G, A', B', C' must be distinct. If, in G, X^* lies outside or of $\Delta A'B'C'$, then one of the curves α, β, γ, combined with the line segment connecting its endpoints, encloses X^*. W.l.o.g. assume this to be α. Therefore, $|\alpha| > |AX^*| + |BX^*|$, and $D_G(A, B) > D_{G*}(A, B)$. Thus, G would be suboptimal. Therefore X^* must lie on the boundary or in the interior of $\Delta A'B'C'$. But then, by Lemma 9, the cost of G equals $|AA'| + |BB'| + |CC'| + |A'B'| + |B'C'| + |A'C'| > |AA'| + |BB'| + |CC'| + |A'X^*| + |B'X^*| + |C'X^*| \geq |AX^*| + |BX^*| + |CX^*| = L^*$. Thus, an optimal spanner G cannot exist. Therefore, for $L = L^*$, the optimal spanner is the network with $A' = B' = C' = X^*$.

Now consider $L < L^*$. Suppose that there exists a weight $L < L^*$ for which an optimal spanner G exists with A', B', C' distinct. Then, by Lemma 8, $D_G = D_G(A, B) = D_G(A, C) = D_G(B, C)$. Since G^* is optimal, and G has weight at most $L < L^*$, we must have $D_G \geq D_{G^*}$. In G^*, the total length of curves α, β, γ equals $D_{G^*} \cdot (|AB| + |BC| + |AC|)$. Since each edge in G^* is contained in exactly two paths, this equals $2L^*$. However, in G, edges $A'B', A'C', B'C'$ are only contained in a single path. Thus, $D_G \cdot (|AB| + |BC| + |AC|) < 2L$. Then, $\frac{2L}{|AB|+|BC|+|AC|} > D_G \geq D_{G*} = \frac{2L^*}{|AB|+|BC|+|AC|}$. This contradicts with the definition of G as an optimal spanner with weight $L < L^*$. Therefore, $A' = B' = C'$ for any optimal spanner with weight $L \leq L^*$. □

Lemma 11 *For any weight L such that $L > L^*$, A', B', C' are distinct in any optimal spanner.*

Proof. By its definition, the dilation star is optimal among spanners satisfying $A' = B' = C'$. However, by Lemma 1, the dilation star cannot be optimal. As such, no spanner satisfying $A' = B' = C'$ is optimal. Then, by Lemma 7, A', B', C' must be distinct in any optimal spanner. □

3 Equilateral triangle

Lemma 12 *In an optimal spanner for an equilateral triangle, $d_G(A, B) = d_G(A, C) = d_G(B, C)$.*

Proof. By Lemma 7, we only need to consider the spanners in which A', B', C' all coincide or are all distinct. As by the definition of the equilateral triangle $|AB| = |AC| = |BC|$, it suffices to show that $D_G(A, B) = D_G(A, C) = D_G(B, C)$ for any optimal spanner.

The case where $A' = B' = C'$ holds then directly follows from Lemma 11 and the fact that the Fermat point and the dilation center coincide for equilateral triangles.

The case where A', B', C' are distinct directly follows from Lemma 8. $\square$

Lemma 13 *Let for a triangle ΔABC the sum of the distances from the vertices to the Fermat point, $|a'| + |b'| + |c'|$, be given. Then the sum of the length of the triangle's edges, $|a| + |b| + |c|$, is minimal for ΔABC equilateral.*

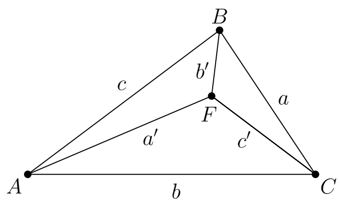

Figure 4: Illustration of notation used in the proof of Lemma 13

Proof. Suppose one of the angles of ΔABC is greater than or equal to 120° in an optimal spanner. Without loss of generality, assume $\angle ABC \geq 120°$ be given. Then, as the Fermat point of ΔABC coincides with point B, $|a'| + |b'| + |c'| = |a| + |c|$. By the law of sines:

$$\frac{|c|}{\sin(\angle ACB)} = \frac{|a|}{\sin(\angle BAC)} = \frac{|b|}{\sin(\angle ABC)} = d$$

where d is the diameter of the circumcircle. Then:

$$\begin{aligned} |b| &= d \cdot \sin(\angle ABC) \\ &= \frac{\sin(\angle ABC)(|a| + |c|)}{\sin(\angle BAC) + \sin(\angle ACB)} \end{aligned}$$

To find the minimum for $|a|+|b|+|c|$ given $|a'|+|b'|+|c'|$, we want to determine the minimum for $|b|$. As $\angle ABC$ and $|a| + |c|$ given, $\angle BAC$ suffices to describe the entire triangle. Thus:

$$\begin{aligned} \frac{d}{d\angle BAC}|b| &= \sin(\angle ABC) \cdot (|a| + |c|) \\ &\cdot \frac{\cos(180° - \angle BAC - \angle ABC) - \cos(\angle BAC)}{(\sin(\angle BAC) + \sin(180° - \angle BAC - \angle ABC))^2} \end{aligned}$$

Equating to zero and solving using the constraints that $120° \leq \angle ABC < 180°$ and that $\angle ABC + \angle ACB + \angle BAC = 180°$, we find an extremum at $\angle BAC = \angle ACB = \frac{180° - \angle ABC}{2}$. As the derivative increases for increasing $\angle BAC$, this must be a minimum. Thus, $|b| \geq \frac{\sqrt{3}}{2}(|a| + |c|) = \frac{\sqrt{3}}{2}(|a'| + |b'| + |c'|)$. Thus, $|a| + |b| + |c| \geq \frac{\sqrt{3}+2}{2}(|a'| + |b'| + |c'|)$

We compare this with ΔABC equilateral:

$$|a| + |b| + |c| = 3|a| = 3\sqrt{3}|a'| = \sqrt{3}(|a'| + |b'| + |c'|)$$

Thus, for $|a'| + |b'| + |c'|$ given, the sum of the length of the triangle's edges is smaller for an equilateral triangle than for a triangle with an angle greater than 120°.

Now assume $\angle BAC \leq 120°$, $\angle ABC \leq 120°$ and $\angle ACB \leq 120°$. Let $S = |a'| + |b'|$ be given. As the Fermat point F is also a Steiner point in the minimum Steiner tree of ΔABC, $\angle AFB = \angle BFC = \angle AFC = 120°$. Then, by the law of cosines:

$$\begin{aligned} |c|^2 &= |a'|^2 + |b'|^2 - 2|a'||b'|\cos(120°) \\ &= |a'|^2 + |b'|^2 + |a'||b'| \\ &= |a'|^2 + (S - |a'|)^2 + |a'|(S - |a'|) \\ &= |a'|^2 + S^2 - |a'|S \end{aligned}$$

As S is fixed, $|c|^2$ is solely dependent on $|a'|$. Therefore:

$$\frac{d|c|^2}{d|a'|} = 2|a'| - S$$

To find the minimum value for $|c|$ given S, we equate the derivative to zero.

$$\begin{aligned} & 2|a'| - S = 0 \\ \iff \quad & 2|a'| = S \\ \iff \quad & 2|a'| = |a'| + |b'| \\ \iff \quad & |a'| = |b'| \end{aligned}$$

Thus, $|c|$ is minimal for ΔABF isosceles. Then from simple trigonometry, utilizing $\angle AFB = 120°$, it follows that in this case, $|c| = \sqrt{3}|a'| = \sqrt{3}\frac{S}{2}$. Thus, more generally, $|c| \geq \sqrt{3}\frac{S}{2} = \sqrt{3}\frac{|a'|+|b'|}{2}$, where $|c| = \sqrt{3}\frac{|a'|+|b'|}{2}$ is only achieved for $|a'| = |b'|$.

Similarly for the other edges, by symmetry:

$$\begin{aligned} |a| &\geq \frac{\sqrt{3}}{2}(|b'| + |c'|) \\ |b| &\geq \frac{\sqrt{3}}{2}(|a'| + |c'|) \end{aligned}$$

Then, for the sum of the triangle's edges:

$$\begin{aligned} |a| + |b| + |c| \geq & \frac{\sqrt{3}}{2}(|b'| + |c'|) + \frac{\sqrt{3}}{2}(|a'| + |c'|) \\ & + \frac{\sqrt{3}}{2}(|a'| + |b'|) \\ = & \sqrt{3}(|a'| + |b'| + |c'|) \end{aligned}$$

As $|a| = \frac{\sqrt{3}}{2}(|b'| + |c'|) \iff |b'| = |c'|$, and as this also holds for $|b|, |c|$ by symmetry, $|a| + |b| + |c| = \sqrt{3}(|a'| + |b'| + |c'|) \iff |a'| = |b'| = |c'|$. Thus for $|a'| + |b'| + |c'|$ given, $|a| + |b| + |c|$ is minimal for ΔABC equilateral. $\square$

Corollary 14 *For the sum of the edge lengths of a triangle ΔABC, $|a|+|b|+|c|$ given, the sum of the distances from the vertices to the Fermat point, $|a'| + |b'| + |c'|$ is maximal for ΔABC equilateral.*

Theorem 15 *Let ΔABC be an equilateral triangle. Then for a given $L \geq \sqrt{3} \cdot |AB|$, the maximal dilation is minimal for the following spanner: an equilateral triangle centered in ΔABC with its vertices connected to the vertices A, B, C such that its vertices lie on the line segments connecting A, B, C to the Fermat point of ΔABC.*

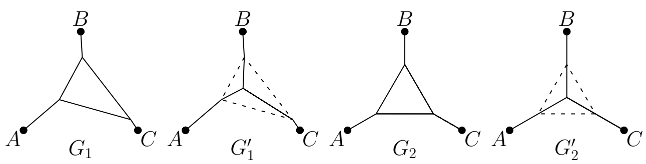

Figure 5: Graphs used in the proof of Theorem 15

Proof. W.l.o.g. assume $|AB| = 1$. Let G_1 be an optimal spanner which does not have this structure. By Lemma 6, it has the general form as depicted in Figure 5. Let s_1 be the sum of the edge lengths of its inner triangle, and t_1 the sum of the lengths of the segments connecting the vertices of the inner triangle and A, B, C. Let D_1 be its maximum dilation, $\max(D_{G_1}(A, B), D_{G_1}(A, C), D_{G_1}(B, C))$, and let L_1 be its total cost.

Then, $L_1 = s_1 + t_1$. Furthermore, as ΔABC equilateral, $|AB| = |AC| = |BC|$, and as by Lemma 12 $d_{G_1}(A, B) = d_{G_1}(A, C) = d_{G_1}(B, C)$, it follows that $D_{G_1}(A, B) = D_{G_1}(A, C) = D_{G_1}(B, C)$. Then:

$$\begin{aligned} 3D_1 &= 3 \cdot \max(D_{G_1}(A, B), D_{G_1}(A, C), D_{G_1}(B, C)) \\ &= D_{G_1}(A, B) + D_{G_1}(A, C) + D_{G_1}(B, C) \\ &= s_1 + 2t_1 \end{aligned}$$

Now consider a spanner G_2, with a centered equilateral triangle with its vertices on the line segments connecting A, B, C to the Fermat point of ΔABC, where the sum of the edge lengths of the inner triangle, s_2, equals s_1. Let t_2 be the sum of the distances between the inner triangle's vertices and A, B, C, L_2 its total cost and D_2 its maximum dilation. Then, as G_1 is an optimal spanner, $L_2 \geq L_1$ or $D_2 \geq D_1$. Next, we consider our previously found equations, utilizing $s_1 = s_2$:

$$\begin{aligned} & L_2 \geq L_1 \\ \iff \quad & s_2 + t_2 \geq s_1 + t_1 \\ \iff \quad & t_2 \geq t_1 \end{aligned}$$

$$\begin{aligned} & D_2 \geq D_1 \\ \iff \quad & 3D_2 \geq 3D_1 \\ \iff \quad & s_2 + 2t_2 \geq s_1 + 2t_1 \\ \iff \quad & t_2 \geq t_1 \end{aligned}$$

Thus, both statements are equivalent to $t_2 \geq t_1$.

Next consider replacing the edges of the inner triangles of G_1 and G_2 by three line segments between the vertices of the inner triangle to its Fermat point. Let these new spanners be G'_1 and G'_2. Let u_1 and u_2 be the sum of the lengths of these three segments for G'_1 and G'_2 respectively. From the total cost of the minimal Steiner tree of ΔABC equaling $\sqrt{3} \cdot |AB|$, it follows that $t_1 + u_1, t_2 + u_2 \geq \sqrt{3} \cdot |AB|$.

First consider the case where the inner triangle of G_1 is equilateral. Then $u_1 = u_2$, but as it is not centered with its vertices on the line segments connecting A, B, C to the Fermat point of ΔABC, $t_1 + u_1 > \sqrt{3} \cdot |AB|$. By construction, G'_2 is the minimum Steiner tree for ΔABC and thus $t_2 + u_2 = \sqrt{3} \cdot |AB|$. But then, as $u_1 = u_2$, $t_1 > t_2$.

Secondly, we consider the case where the inner triangle of G_1 is not equilateral. As G'_2 is the minimum Steiner tree for ΔABC, $t_2 + u_2 = \sqrt{3} \cdot |AB|$. By Corollary 14 $u_1 < u_2$. But then, as $t_1 + u_1 \geq \sqrt{3} \cdot |AB|$, $t_1 > t_2$.

Thus in all cases $t_1 > t_2$. Thus G_1 cannot be optimal. □

Theorem 16 *Let points A, B, C, the vertices of an equilateral triangle, and $\sqrt{3} \cdot |AB| \leq L \leq 3 \cdot |AB|$, the maximum total cost, be given. Then, in an optimal spanner,*

$$\begin{aligned} A' &= A + \frac{\vec{AB} + \vec{AC}}{|\vec{AB} + \vec{AC}|} \cdot \frac{3 \cdot l_o - L}{3\sqrt{3} - 3}, \\ B' &= B + \frac{\vec{BA} + \vec{BC}}{|\vec{BA} + \vec{BC}|} \cdot \frac{3 \cdot l_o - L}{3\sqrt{3} - 3}, \\ C' &= C + \frac{\vec{CA} + \vec{CB}}{|\vec{CA} + \vec{CB}|} \cdot \frac{3 \cdot l_o - L}{3\sqrt{3} - 3} \end{aligned}$$

with $l_o = |AB|$.

See appendix for the proof of Theorem 16.

4 Isosceles triangle

Let ΔABC be an isosceles triangle with $|AB| = |BC|$. We analyze the case of small L, that is for $L \leq L^*$, for which we know by Lemma 10 that $A' = B' = C'$.

Lemma 17 *In a spanner with $X := A' = B' = C'$, for X at a given height with respect to base AC, $|AX| + |CX|$ is minimal for $|AX| = |CX|$.*

See appendix for the proof of Lemma 17.

Corollary 18 *In an optimal spanner, if $X := A' = B' = C'$, this point lies on the perpendicular bisector of AC.*

Proof. Let G be a given optimal spanner, in which $|AX| \neq |CX|$. Without loss of generality due to the symmetry of an isosceles triangle, assume $D_G(A,B) \geq D_G(B,C)$. We will consider a spanner G^*, in which $|AX^*| = |CX^*|$, but where $|X^*X'^*|$ equals $|XX'|$ in G, where X' is the perpendicular projection of X on AC. Then, by applying Lemma 17 and the Pythagorean Theorem to $|BX|$, it follows that the total cost of G^* is smaller than the total cost of G.

We further show that the maximal dilation is reduced. As $|AX| + |CX|$ minimal for $|AX| = |CX|$, $D_G(A,C) > D_{G^*}(A,C)$. By the Pythagorean Theorem, $|AX^*| < |AX|$ and $|BX^*| < |BX|$. But then, $D_G(A,B) > D_{G^*}(A,B) = D_{G^*}(B,C)$. But then, G cannot be optimal, and thus for any optimal spanner, $|AX| = |CX|$ must hold. □

Together with Lemma 1 and Lemma 9 this provides a complete characterization of the spanner for the case that $L_{MST} \leq L \leq L^*$, where L_{MST} is the weight of the minimum Steiner tree and L^* is the weight of the minimum dilation star. We have not yet found similar characterizations for larger L. Conjectures based on computational experiments are included in Section 5.

5 Computational experimentation

To further investigate properties of optimal spanners, we used an approximation algorithm to find such spanners for various problem instances. We chose not to use exact algorithms like those developed by Kooijmans [13] and Verstege [18] due to the simplicity of the considered cases. Consequently, simple approximation algorithms, such as the evolution-based approach we implemented, give satisfactory results while avoiding error-prone complex implementations.

In Figure 6, for several problem instances and for increasing maximum weight, the computationally found approximations are visualized. From the development of the spanner for increasing maximum weight, we deduce multiple conjectures.

Conjecture 1 *Let ΔABC be an isosceles triangle with $|AB| = |BC|$. Then the optimal spanner is symmetric in the perpendicular bisector of AC.*

Conjecture 2 *Let ΔABC be an isosceles triangle with $|AB| = |BC|$. If $\angle ABC \leq 120°$, $|AA'| > 0, |BB'| > 0, |CC'| > 0$ in the optimal spanner for $|AF| + |BF| + |CF| < L < |AB| + |AC| + |BC|$ with F the Fermat point of ΔABC.*

Note that in Conjecture 2, the lower and upper bounds on L are given by the cost of the minimum Steiner tree and the complete graph respectively.

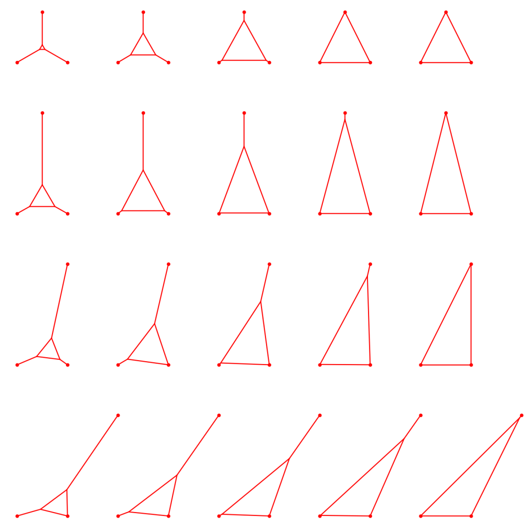

Figure 6: Approximately optimal spanners for various problem instances of three points at varying maximum weight

Conjecture 3 *Let L^* be the weight of the dilation star. Then $\lim_{L \downarrow L^*} \angle A'B'C' = \lim_{L \downarrow L^*} \angle A'C'B' = \lim_{L \downarrow L^*} \angle B'A'C' = 60°$.*

Conjecture 4 *If in an optimal spanner $|AA'|, |BB'|, |CC'| > 0$, the lines AA', BB', CC' intersect in a single point.*

6 Conclusion

The aim of our work was to analyze the geometry of solutions to the Restricted-Weight Minimum-Dilation Spanner Problem with Steiner Points. As presented, we have determined the topology of solutions for instances with $|P| = 3$. Additionally, we can fully describe the optimal spanner if the points in P are the vertices of an equilateral triangle, and partially (i.e. if the maximum weight is small) if they form an isosceles triangle.

We leave multiple open problems. Proofs for optimal spanners for isosceles triangles and larger weight are still elusive. Similarly, descriptions of optimal spanners and corresponding proofs for general triangles, quadrilaterals (in particular the square), pentagons and further polygons are yet to be found. In restricted settings, the previous work by Kooijmans [13] and Verstege [18] provides insights into the topologies of solutions. Considering optimal spanners more generally, related problems include solutions and their structure for other objective functions in weight-restricted settings, for example, minimizing the diameter or the number of edges.

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Appendix

Proof for Lemma 1

Proof. Assume there exists an optimal spanner where the maximal dilation is greater than 1, and the total cost is not maximal. Let $L_u < L$ be the total weight of the optimal spanner.

Firstly, we will consider the case that a single path has maximal dilation. W.l.o.g. assume that α is said path. Thus, $D_G(A,B) > D_G(B,C)$, $D_G(A,B) > D_G(A,C)$. Then, as $D_G(A,B) > 1$, α is not a single line segment. Then we can use the available weight $L - L_u$ to decrease $d_G(A,B)$, thereby decreasing $D_G(A,B)$. Thus the solution is not optimal.

Next, we consider the case that two paths have maximal dilation. W.l.o.g. assume that α, β are said paths. Thus, $D_G(A,B) = D_G(B,C) > D_G(A,C)$. Then, as $D_G(A,B) = D_G(B,C) > 1$, neither α nor β is a single line segment. Then we can use half of the available weight $L - L_u$ to decrease $d_G(A,B)$ and half to decrease $d_G(B,C)$, thereby decreasing $D_G(A,B), D_G(B,C)$. Thus the solution is not optimal.

Finally, we consider the case where $D_G(A,B) = D_G(B,C) = D_G(A,C)$. Then, as $D_G(A,B) = D_G(B,C) = D_G(A,C) > 1$, none of the paths consist of a single line segments. Thus we can use a third of the available weight $L - L_u$ for decreasing $d_G(A,B), d_G(B,C), d_G(A,C)$ each respectively, thereby decreasing the maximal dilation. Thus the solution is not optimal.

Thus in all cases, the presumed solution is not optimal. □

Proof for Lemma 2

Proof. W.l.o.g. we will consider α. Let α not be injective. Then there thus exist $0 \leq x < y \leq 1$ such that $\alpha(x) = \alpha(y)$.

If $\alpha[x,y] \setminus (\beta \cup \gamma) \neq \emptyset$, redefining α to the concatenation of $\alpha[0,x]$ and $\alpha[y,1]$, will decrease the total cost while not increasing the maximum dilation. Thus by Lemma 1, the solution cannot be optimal.

Otherwise, $(\beta \cup \gamma) \cap \alpha[x,y] = \alpha[x,y]$. Then let $b_j = \inf\{b : \beta(b) \in \alpha[x,y]\}, c_j = \inf\{c : \gamma(c) \in \alpha[x,y]\}$ be the indices at which β, γ join $\alpha[x,y]$ respectively, and let $b_l = \sup\{b : \beta(b) \in \alpha[x,y]\}, c_l = \sup\{c : \gamma(c) \in \alpha[x,y]\}$ be the indices at which they leave $\alpha[x,y]$. If $|\beta[b_j, b_l]| > \frac{|\alpha[x,y]|}{2}$, we can redefine $\beta[b_j, b_l]$ to $\alpha[x,y] \setminus \beta[b_j, b_l]$, resulting in $|\beta[b_j, b_l]| \leq \frac{|\alpha[x,y]|}{2}$ without increasing dilation. Analogously, we can redefine $\gamma[c_j, c_l]$ such that $|\gamma[c_j, c_l]| \leq \frac{|\alpha[x,y]|}{2}$. Then, if $\alpha[x,y] \setminus (\beta \cup \gamma) \neq \emptyset$ for the redefined curves, the modified solution, and therefore the original solution, cannot be optimal as previously described. Otherwise, $|\beta[b_j, b_l]| = |\gamma[c_j, c_l]| = \frac{|\alpha[x,y]|}{2}$, and we can redefine $\beta[b_j, b_l]$ to $\gamma[c_j, c_l]$ without increasing dilation. Then $\alpha[x,y] \setminus (\beta \cup \gamma) \neq \emptyset$, and thus the modified solution, and therefore the original solution cannot be optimal as previously shown.

Thus, in all cases the solution is not optimal, and we conclude that α must be injective in an optimal solution. □

Proof for Lemma 3

Proof. Let X, Y be points shared by curves α, β. Assume that $\alpha[X \to Y] \neq \beta[X,Y]$ in an optimal solution. Then $\alpha[X,Y] \setminus \beta[X,Y] \neq \emptyset$ and $\beta[X,Y] \setminus \alpha[X,Y] \neq \emptyset$.

Suppose that $|\alpha[X,Y] \setminus \beta[X,Y]| < |\beta[X,Y] \setminus \alpha[X,Y]|$. Then we can redefine $\beta[X,Y]$ to $\alpha[X,Y]$ without increasing dilation, while maintaining or lowering the total cost. The cost is only maintained if $\beta[X,Y] \setminus \alpha[X,Y] \subseteq \gamma$. Then, $\gamma \cap (\beta[X,Y] \setminus \alpha[X,Y])$ can also be redefined to segments of $\alpha[X,Y]$ without increasing the dilation, while reducing the cost. By Lemma 1, the modified solution, and therefore the original solution, cannot be optimal. Thus, $|\alpha[X,Y] \setminus \beta[X,Y]| \geq |\beta[X,Y] \setminus \alpha[X,Y]|$. Analogously, $|\alpha[X,Y] \setminus \beta[X,Y]| \leq |\beta[X,Y] \setminus \alpha[X,Y]|$.

Thus, $|\alpha[X,Y] \setminus \beta[X,Y]| = |\beta[X,Y] \setminus \alpha[X,Y]|$. Then, as previously shown, we can redefine $\beta[X,Y] \setminus \alpha[X,Y]$, and possibly $\gamma \cap (\beta[X,Y] \setminus \alpha[X,Y])$, to $\alpha[X,Y]$ to reduce the total cost without increasing the dilation. As such, by Lemma 1, the modified solution, and therefore the original solution, cannot be optimal.

Analogously for pairs α, γ and β, γ. □

Proof of Lemma 4

Proof. If $L = |AB| + |AC| + |BC|$, the optimal solution is given by the complete graph, which consists of straight line segments.

If $L < |AB| + |AC| + |BC|$, w.l.o.g. let us consider α. Let α contain a non-straight arc in an optimal solution. Let $\alpha[x,y]$ be such an arc, with $0 < x < y < 1$.

If $\alpha[x,y] \cap \beta = \alpha[x,y] \cap \gamma = \emptyset$, replacing $\alpha[x,y]$ with a direct line segment decreases cost, while not increasing dilation. By Lemma 1, the modified solution, and therefore the original solution cannot be optimal.

If $\alpha[x,y] \cap \beta \cap \gamma = \alpha[x,y]$, α, β, γ can all be redefined to a direct line segment to decrease cost, while not increasing dilation. By Lemma 1, the modified solution, and therefore the original solution cannot be optimal.

If $\alpha[x,y] \cap \beta = \alpha[x,y]$ and $\alpha[x,y] \cap \gamma = \emptyset$, both α, γ can be redefined to a direct line segment to decrease the cost, while not increasing dilation. By Lemma 1, the modified solution, and therefore the original solution cannot be optimal. By similar argument, the same holds for $\alpha[x,y] \cap \beta = \emptyset$ and $\alpha[x,y] \cap \gamma = \alpha[x,y]$.

Otherwise, we can introduce $z_1 < z_2 < \ldots < z_n$, with $z_1 > x, z_n < y$, such that $\alpha[x, z_1], \alpha[z_1, z_2], \alpha[z_2, z_3], \ldots, \alpha[z_{n-1}, z_n], \alpha[z_n, y]$ all either reflect one of the previous cases or are a straight line segment.

Thus, in all cases, non-straight arcs result in a non-optimal solution. □

Proof for Lemma 5

Proof. By Lemma 4, we can consider the solution to be a network over a set of points V, consisting of the triangle's vertices and additional Steiner points.

Let $h_a(p)$ for $p \in V$ denote the distance from p to the line BC if the line segment connecting p and A intersects the line BC, and the additive inverse of the distance from p to the line BC otherwise.

Assume that there exists a point q such that $h_a(q)$ positive in an optimal solution. Then let $q_1, q_2, \ldots, q_n \in V$ denote the points with $h_a(q_1) = h_a(q_2) = \ldots = h_a(q_n) = \max_{p\in V} h_a(p)$. Now move all q_i for $1 \leq i \leq n$ perpendicular to the line BC by ϵ towards said line, for ϵ sufficiently small. Then any line segment between q_i, q_j for $1 \leq i < j \leq n$, is not changed in length. For segments between p', q', where $h_a(p') < h_a(q') = \max_{p\in V} h_a(p)$, by the Pythagorean Theorem, the length is reduced. All remaining segments are not affected.

As $h_a(q) > 0$, and as each point in V is (in)directly connected to A, B, C, there must exist p', q' such that $h_a(p') < h_a(q') = \max_{p\in V} h_a(p)$ and p', q' directly connected. Thus, there must exist a segment which has been shortened, while none of the segments has increased in length. As such, the modified solution has a lower cost. By Lemma 1, the modified solution can thus not be optimal. Furthermore, as the maximum dilation has thus also not increased, the original solution can also not be optimal. As such, our assumption that there exists a point q such that $h_a(q) > 0$ cannot hold.

Analogously, an optimal solution cannot contain a point q such that the line segment connecting q and B intersects the line AC, or such that the line segment connecting q and C intersects the line AB. $\square$

Proof for Lemma 9

Proof. For M in the interior of ΔABC, we trivially have $|AM| + |MB| < |AC| + |CB|$. Similarly, $|BM| + |MC| < |BA| + |AC|$ and $|AM| + |MC| < |AB| + |BC|$. Adding these equations gives $2\cdot(|AM|+|BM|+|CM|) < 2\cdot(|AB|+|BC|+|AC|)$.

For M on the boundary of ΔABC, $|AM|+|MB| \geq |AC|+|CB|$ if and only if $M = C$. Similarly, $|BM|+|MC| \geq |BA|+|AC|$ if and only if $M = A$ and $|AM|+|MC| \geq |AB|+|BC|$ if and only if $M = B$. Thus, as A, B, C are distinct, for any point on the boundary, $2\cdot(|AM| + |BM| + |CM|) < 2\cdot(|AB| + |BC| + |AC|)$ must hold. $\square$

Proof for Theorem 16

Proof. Let points A, B, C, the vertices of an equilateral triangle, and L, the maximum total cost, be given. We will now construct A', B', C'. Let $l_o := |AB| = |AC| = |BC|$. Furthermore, let $l_c := |AA'|$, and let $l_i := |A'B'|$. Then, by Theorem 15, $l_c = |AA'| = |BB'| = |CC'|$ and $l_i = |A'B'| = |A'C'| = |B'C'|$.

Then, as by Theorem 15 $\angle A'AC = 30°$, $l_o = 2\cdot l_c \cdot \cos(30°) + l_i$. From this equation, it follows that $l_i = l_o - 2\cdot l_c\cdot\cos(30°)$. Furthermore, by Lemma 1, $3\cdot l_c + 3\cdot l_i = L$. Then we can solve for l_c:

$$\begin{aligned}
& 3\cdot l_c + 3\cdot l_i = L \\
\iff\quad & 3\cdot l_o - 6\cdot l_c\cdot\cos(30°) + 3\cdot l_c = L \\
\iff\quad & 3\cdot l_o - 3\sqrt{3}\cdot l_c + 3\cdot l_c = L \\
\iff\quad & (3-3\sqrt{3})\cdot l_c = L - 3\cdot l_o \\
\iff\quad & l_c = \frac{3\cdot l_o - L}{3\sqrt{3}-3}
\end{aligned}$$

Finally, as by Theorem 15 $\angle A'AB = \angle A'AC$ and $|AB| = |AC|$:

$$A' = A + \frac{\vec{AB}+\vec{AC}}{|\vec{AB}+\vec{AC}|}\cdot\frac{3\cdot l_o - L}{3\sqrt{3}-3}$$

And by symmetry analogously for B' and C'. $\square$

Proof for Lemma 17

Proof. Define X' as the perpendicular projection of X on AC, such that $\angle AX'X = 90°$, which is in any case valid by Lemma 5. Now let $|XX'|$ be fixed. Then:

$$\begin{aligned}
|AX| + |CX| &= \sqrt{|AX'|^2 + |XX'|^2} + \sqrt{|CX'|^2 + |XX'|^2} \\
&= \sqrt{|AX'|^2 + |XX'|^2} \\
&\quad + \sqrt{(|AC| - |AX'|)^2 + |XX'|^2}
\end{aligned}$$

To find the minimum, we differentiate with respect to $|AX'|$:

$$\begin{aligned}
\frac{d(|AX| + |CX|)}{d|AX'|} &= \frac{|AX'|}{\sqrt{|AX'|^2 + |XX'|^2}} \\
&\quad - \frac{|AC| - |AX'|}{\sqrt{(|AC| - |AX'|)^2 + |XX'|^2}} \\
&= \frac{|AX'|}{\sqrt{|AX'|^2 + |XX'|^2}} \\
&\quad - \frac{|CX'|}{\sqrt{|CX'|^2 + |XX'|^2}}
\end{aligned}$$

Setting the derivative equal to zero, we find:

$$\begin{aligned}
& \frac{d(|AX| + |CX|)}{d|AX'|} = 0 \\
\iff\quad & \frac{|AX'|}{\sqrt{|AX'|^2 + |XX'|^2}} = \frac{|CX'|}{\sqrt{|CX'|^2 + |XX'|^2}} \\
\iff\quad & |AX'|^2(|CX'|^2 + |XX'|^2) = |CX'|^2(|AX'|^2 + |XX'|^2) \\
\iff\quad & |AX'|^2|XX'|^2 = |CX'|^2|XX'|^2 \\
\iff\quad & |AX'|^2 = |CX'|^2
\end{aligned}$$

And as lengths of line segments are non-negative, $|AX'| = |CX'|$. But then, by the Pythagorean Theorem, $|AX| = |CX|$. $\square$