A lower bound on the number of colours needed to nicely colour a sphere

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Abstract

The Hadwiger–Nelson problem is about determining the chromatic number of the plane (CNP), defined as the minimum number of colours needed to colour the plane so that no two points of distance 1 have the same colour. In this paper we investigate a related problem for spheres and we use a few natural restrictions on the colouring. Thomassen showed that with these restrictions, the chromatic number of all manifolds satisfying certain properties (including the plane and all spheres with a large enough radius) is at least 7. We prove that with these restrictions, the chromatic number of any sphere with a large enough radius is at least 8. This also gives a new lower bound for the minimum colours needed for colouring the 3-dimensional space with the same restrictions.

1 Introduction

1.1 Colourings of the plane

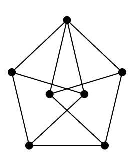

Figure 1

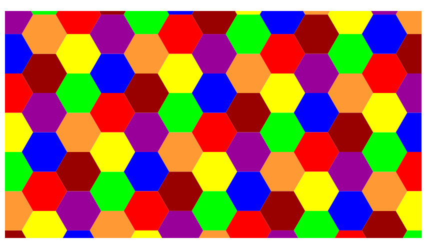

Figure 2

The Hadwiger–Nelson problem is a well-known problem in combinatorial geometry. It asks to determine the chromatic number of the plane (CNP), i.e., the minimum number of colours needed to colour the plane so that no two points of distance 1 have the same colour. Alternatively, it is the chromatic number of the graph of unit distances on the plane. Since 1950 it has been known that $4 \leq CNP \leq 7$. The lower bound was obtained by Nelson (1950), but it can be most easily proven by using a graph called the Moser spindle (Figure 1) (Moser, Moser (1961) [M]), while the upper bound was given by Isbell (1950), using the colouring in Figure 2. Since 2018, it is also known that $CNP \geq 5$ (de Grey [dG]).

The problem has some variations:

If we restrict the colour classes to be measurable, the best known lower bound for the number of colours needed is also 5 (Falconer (1981) [F]) and the best known upper bound is also 7 (also from Figure 2).

If we restrict the colour classes to be the unions of shapes bounded by Jordan curves (such a shape is called a tile and such a colouring is called a tile-based colouring or simply a tiling), then the best known lower bound for the number of colours needed is 6 (Townsend (2005) [T]) and the best known upper bound is also 7.

Thomassen also defined a type of tiling:

A colouring of a surface with a metric is *nice*, if it is a tiling, all tiles have diameter less than 1, all pairs of tiles with the same colour have distance more than 1 and all tiles are simply connected. We refer to such a colouring as a nice tiling.

He also proved the following theorem:

Theorem 1 *[T] Suppose a surface S satisfies the following three conditions for some natural number k:*

1. Every noncontractible simple closed curve has diameter at least 2.

2. If C is a simple closed curve of diameter less than 2, then the area of $int(C)$ is at most k.

3. The diameter of S is at least $12k + 30$.

Then every nice tiling contains at least 7 colours.

Since the plane satisfies the conditions, the theorem proves that every nice tiling of the plane contains at least 7 colours.

Note that the statement for the plane also follows from a relatively easy proof using Lemma 2 and the fact that a triangulated planar graph has $3n - 6$ edges.

1.2 Colouring of spheres

We can define the chromatic number of a sphere of radius r similarly to the planar case: it is the minimum number of colours needed to colour the points of a sphere of radius r such that no two points with Euclidean distance 1 have the same colour.

Much less is known of the value of this number compared to the planar case.

It is known that the chromatic number of a sphere of radius r is at least 4 if $r \geq \frac{1}{\sqrt{3}}$. For $r > \frac{\sqrt{3}}{2}$ Moser's

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spindle gives the lower bound, for smaller values, a generalized version of Moser's spindle is used. (Simmons (1976) [S])

It is also known that the chromatic number of any sphere is at most 15, even with all of the above defined restrictions, as the 3-dimensional space has a 15-tiling (Radoičić, Tóth (2003)[RT]), which can be used to generate such a colouring.

Recently, a 7-colouring of large enough spheres have also been found by Tom Sirgedas, as part of the Polymath 16 project.[1]

Also, the minimal number of colours needed for a nice tiling of a large enough sphere is at least 7, which follows from Theorem 1.

The main result of this paper is improving this number to 8.

Note that some sources mentioned earlier that the chromatic number of all spheres is at most 7 [BMP] [HDCG]. It seems (from personal communication through Dömötör Pálvölgyi) that the authors expected that a colouring similar to that of Isbell in Figure 2 also works for spheres. The present paper disproves this assumption, though it does not contradict to the chromatic number of spheres being at most 7.

This result also improves the lower bound for the minimal number of colours needed for a nice tiling of the 3-dimensional space, for which problem the best known bound was 6 for the general colouring case (Nechushtan (2002) [N]).

2 Results

2.1 Preliminary statements

Definition 1 *We call a colouring of a graph nice, if there are no two different vertices within distance at most* 2*, which are coloured with the same colour. Alternatively, a nice colouring can be defined as a colouring of* G*, which is also a proper colouring for* G^2 *(the square of* G*).*

Lemma 2 *If a tiling of a surface is nice, then applying the same colouring to the adjacency graph of the tiles also gives a nice colouring.*

The proof is in the Appendix.

Lemma 3 *If* S *is a sphere with radius* $r \geq \frac{2}{\pi}$ *and* $A \subseteq S$ *such that the spherical diameter of* A *is less than* 1*, then there is a unique connected component of* $S \setminus A$*, which contains all points of* S *with the exception of at most an open disk of radius* 1*.*

The proof is in the Appendix.

[1] https://groups.google.com/forum/#!topic/hadwiger-nelson-problem/tSOs7MypGxE

Lemma 4 *If* S *is a sphere with radius larger than* $\frac{2}{\pi}$ *and the adjacent tiles* $A, B \subseteq S$ *both have spherical diameter less than* 1*, then there is a unique connected component of* $S \setminus (A \cup B)$*, which contains all points of* S *with the exception of at most an open disk of radius* 1*.*

The proof is in the Appendix.

For any set $A \subseteq S$ or two sets $A, B \subseteq S$, call the unique component described above the large component of $S \setminus A$ or $S \setminus (A \cup B)$, respectively, and all the other components the small components.

2.2 The main result

Theorem 5 *There is no nice tiling with at most* 7 *colours of a large enough (radius* $r \geq 18$*) sphere* S*, even if we generalize the definition and allow tiles not to be simply connected.*

In order to make the proof more legible, we give an outline of the main steps.

Suppose that there exists such a colouring of S and take the adjacency graph G of the tiles such that all tiles are represented by one of their points. First, by deleting some vertices and edges, we get rid of all multiple edges and cut vertices and get a graph G', which is a triangulated planar graph and still has the property that all of its pairs of neighbouring vertices have distance less than 2. If we had a nice tiling of S, we also have a colouring of this graph, which is not only nice, but also no two vertices with distance at most 1 get the same colour. This will lead to a contradiction.

Since G' is a triangulated planar graph with maximum degree at most 6 (otherwise its colouring could not be nice), it has at most 12 vertices with degree less than 6 (exactly 12 if counted with multiplicity given by the differences of 6 and the degrees of these vertices). These vertices are called irregular vertices.

Also, for some subsets of G' that only have vertices with degree 6, there exists a function to an infinite triangular grid such that the mapping is a local isomorphism in all vertices.

Now we have three cases.

The first case is when all of the irregular vertices are close to each other. In this case, we find a cycle c_1 of bounded length separating them from most of S. And from the mapping we can get from the latter part to the triangular grid, we can prove that this part has a bounded graph size. So we can get to a contradiction by finding a vertex far away from the irregular vertices, which exists if r is large enough.

In the second case the irregular vertices can be divided into two groups both of cardinality 6 (counted with multiplicity), where the elements of the two groups have a large enough distance from each other, while inside one group, the distances are bounded. In this case, we can construct two cycles c_2 and c_3 of bounded length separating these two groups, which are close enough to the first and the second group, respectively. Then again we get a contradiction from the mapping of the part between the two cycles to the triangular grid: we find a cycle in this part that goes through two nearly antipodal points, but its graph length is not larger than $max(l(c_2), l(c_3))$.

Finally, the last case is when there is at least one way to divide the irregular vertices into two groups such that no two points from different groups are close to each other and the cardinality of the two groups (counted with multiplicity) is not divisible by 6. In this case, we get a contradiction by examining the exact way to colour parts of the infinite triangular grid.

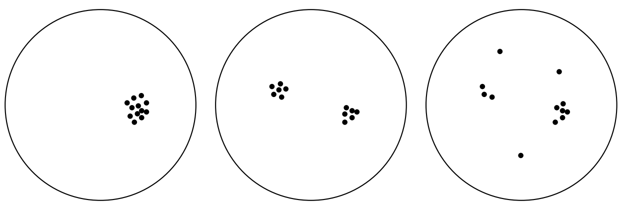

Figure 3: Case 1, Case 2 and Case 3

Now we continue with a detailed proof.

Proof. For the sake of simplicity, first we will use spherical distances in the calculations (so we are now solving the problem when tiles have spherical diameter at most 1 and tiles of the same colour have spherical distance more than 1) and we will only convert the problem to the Euclidean distance definition in the end. So now suppose that S has a radius $r \geq 17.9$.

Suppose we have a nice tiling of S.

Take the graph G where the tiles $T_1, ...T_n$ are represented by vertices $v_1, ...v_n$ and the neighbouring ones are connected. The colouring of the tiles gives a nice colouring of this graph by Lemma 2, which we will use in the proof.

We can also get rid of vertices or edges, and prove the statement for the remaining subgraph as it also implies that the original graph cannot be coloured with a nice colouring with 7 or less colours either. First, eliminate the multiple edges (a multiple edge can occur if two tiles have a border made out of disjoint segments): if we find a pair of parallel edges between v_i and v_j, we can just merge them and delete everything between them (between meaning the vertex set corresponding to the small components of $S \setminus (T_i \cup T_j)$). We also eliminate cut vertices (these correspond to not simply connected tiles) by deleting all vertices corresponding to the tiles in the small components of $S \setminus v_i$ for any cut vertex v_i. Also, we can eliminate those complete graphs with more than 3 vertices that represent points where more than 3 tiles meet: we just take an arbitrary triangulation of them as if the tiles would not exactly meet in one point. This way we have got a triangulated planar graph G', and from now on, triangles will always mean the empty 3-cycles of G'. If G' has n' vertices, it has $3n' - 6$ edges, which means that the sum of the degrees of its vertices is $6n' - 12$. And since there are no vertices with neighbours of the same colour (the colouring for G' is also nice), all vertices have degree at most 6. So there are at most 12 vertices having degree less than 6. Call them irregular vertices and for any irregular vertex, let its multiplicity be the difference of its degree from 6. From the above, there are exactly 12 irregular vertices, if we count them with multiplicity. Also, call the set of irregular vertices I and call the elements of $V(G') \setminus I$ regular vertices.

Now draw G' on S so that $v_i \in T_i$ and the edges are represented by simple Jordan curves satisfying the following conditions:

1) The two endpoints of the image of an edge e has the two vertices e is incident to as its endpoints.

2) If the border of two tiles T_i and T_j contains more than one point, then draw the edge between v_i and v_j so that it only contains points from these two tiles. This also means that all points of the edge have distance less than 2 from both v_i and v_j.

3) For any point in which more than three tiles meet, the edges corresponding to pairs of tiles which only border each other in this point only run through border segments starting from the meeting point. Also, the edges should run so close to the common border point of the tiles that all points of all edges have distance less than 2 (measured on S) from both of the endpoints of that particular edge.

Lemma 6 *Any (open or closed) disk D on S with radius at least 1 contains at least one vertex from G'.*

The proof is in the Appendix.

So it is enough to prove the following (stronger) version of Theorem 5:

Suppose we have a sphere S with radius $r \geq 17.9$ and a fully triangulated planar graph G' on the surface of S, which has n' vertices, all of its vertices have distance less than 2 on S, all open unit disks on S contain at least one vertex and all of the points of all of its edges have distance less than 2 from both of its respective endpoints. Then G' cannot be coloured with 7 colours in a way so that any two vertices of the same colour have graph distance at least 3 in G'.

Now continue with some definitions:

For any two subsets of S, let their *spherical distance* be their spherical distance on S (denoted by $dist_S(a, b)$).

For any two subgraphs G_a and G_b of G', let their *graph distance* mean the smallest graph distance in G' occuring between their vertices (denoted by $dist_{G'}(a, b)$).

For any 3-cycle c in G' that has a side, which is

Let the *graph length* of a path or closed path p in G' be the number of its edges. We denote it by $l(p)$.

Let the *spherical broken line* of a path or closed path p in G' be the curve defined by connecting neighbouring vertices of p with spherical segments instead of the edges connecting them.

Let the *broken line length* of a path or a closed path p in G' be the length of the spherical broken line belonging to p. We denote it by $L(p)$.

Let the *i-neighbourhood* of a vertex v of G' be the subgraph of G' induced by those vertices, which have graph distance at most i from v (denoted by $N_i(v)$).

Let the *strict i-neighbourhood* of a vertex v of G' be the subgraph of G' induced by those vertices, which have graph distance exactly i from v (denoted by $n_i(v)$).

For a vertex v and a set S of vertices, let $e(v, S)$ mean the number of edges starting from v and ending in any of the vertices of S.

We will use the following lemmas later in the proof:

Lemma 7 *For any path or closed path p in G', $L(p) < 2l(p)$.*

Proof. All edges connect points in neighbouring tiles meaning that they have a distance less than 2 as both of the endpoints have a distance at most 1 from an arbitrarily chosen border point. And by summing these inequalities, we get the statement. □

Lemma 8 *If we take a subgraph G^* of G', which does not include any irregular vertices and is defined by a simply connected region S^* on S such that those vertices are included, which are inside S^* and those edges are included, which are fully inside S^*, then there exists a function φ from the vertices and edges of G^* to an infinite triangular grid T (for the sake of simplicity, suppose that it is made up of regular triangles) fulfilling the following criteria:*

1) It keeps the incidence relation between vertices and edges.

2) For any vertex $v \in G^*$, if for two edges e_1 and e_2 in G^*, that have v as an endpoint, there are exactly k edges of G' between them going around v in a positive order, then there are exactly k edges of T between $\varphi(e_1)$ and $\varphi(e_2)$ going in a positive order around $\varphi(v)$.

The above colouring is unique up to isometries preserving orientation.

The proof is in the Appendix.

Lemma 9 *We can find a similar φ function if G^* is defined by a subset S' of S that is homeomorphic to $S^1 \times [0, 1]$ (like S minus two disjoint disks) and still does not contain irregular vertices and we also require G^* to be connected, but here the codomain of φ will be the set of (possibly infinite) sets of vertices in case of vertices and the set of (possibly infinite) sets of edges for edges. Here we require from φ that for any vertex $v \in G^*$, and an edge $e \in G^*$ incident with v all of the elements of $\varphi(v)$ are incident with at least one element of $\varphi(e)$. We also require that for any vertex $v \in H$ and any element $v' \in \varphi(v)$, we can choose an element from all the φ's of the edges incident to v such that they are all incident to v' and for any two of them, they have exactly as many edges between them going around v' in a positive order as the corresponding edges in G^* have going around v in a positive order.*

The proof is in the Appendix.

Analogously to the colouring of the plane by Isbell, call a colouring of the vertices of the infinite triangular grid T an *Isbell colouring* if it is constructed in the following way:

We take a vertex in the grid and colour it and its neighbours with 7 different colours. We then tile the grid with the disjoint translates of this coloured hexagon.

Such a colouring is trivially nice and periodical, thus any Isbell colouring of T can be generated using any of the vertices of T as the starting vertex. Also, for all colourings of the starting hexagon, there are exactly two ways to colour T depending on how we place the hexagons compared to each other. Also, all Isbell colourings can be generated with the above procedure starting from any hexagon formed by a vertex and its 6 neighbours.

Lemma 10 *The graph in Figure 4 can only be nicely 7-coloured by a part of an Isbell colouring.*

The proof is in the Appendix.

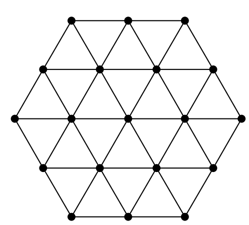

Figure 4

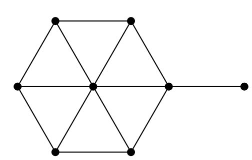

Figure 5

Lemma 11 *If we embed the graph in Figure 5 in the infinite triangular grid, then any colouring of it is contained in at most one Isbell colouring of T.*

Proof. The hexagon part determines the colouring of that particular hexagon, while the remaining vertex leaves at most one of the two colourings that can be generated from that particular colouring of the hexagon. □

Definition 2 *If we have a cycle c in G' then let one of its sides be called as the inside and the set of vertices of G' in it be called V_i, while the set of edges in it be called E_i, while the other one being the outside and call the set of vertices of G' in it V_o and the set of edges in it E_o. The curvature of c is $\sum_{v \in c} 2 - e(v, V_i)$.*

(Note that if c does not contain irregular vertices, then this definition is trivially equivalent with $\sum_{v \in c} e(v, c \cup V_o) - 4$.)

Lemma 12 *The curvature of a cycle c is equal to 6 minus the number of irregular vertices in the inside of c (counted with multiplicity).*

The proof is in the Appendix.

Let H be the graph with vertex set I and edges connecting the pairs of vertices, which have graph distance at most 3 in G'.

Now take a connected component H_i of H and take a spanning tree of H_i. For any edge e of this spanning tree find a path of length at most 3 in G' connecting the two endpoints of e (per definition, such a path exists) and take the union of these paths, which is a graph in G'. Take a spanning tree of this graph and call it H_i'. Do this for all components of H and call the union of these trees H'.

Lemma 13 *The H_i''s have graph distance at least 2 from each other in G'.*

The proof is in the Appendix.

Lemma 14 *If the number of vertices in H_i (counted with multiplicity) is n_i, then $|V(H_i')| \leq 3n_i - 2$, $|E(H_i')| = |V(H_i')| - 1 \leq 3n_i - 3$ and $\{v \in G' | dist_{G'}(v, H_i) = 1\} \leq 5n_i$*

The proof is in the Appendix.

Now take an H_i and take the union U_i of the triangles (borders included) that have at least one vertex in common with H_i.

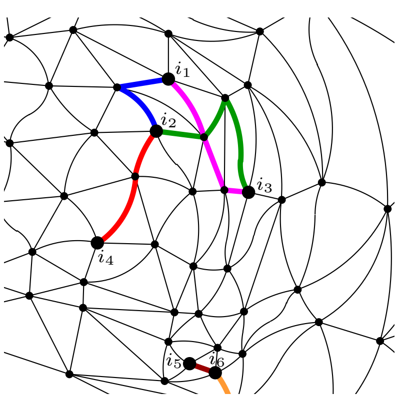

Figure 6: A part of G' with the chosen paths connecting the vertices of I highlighted

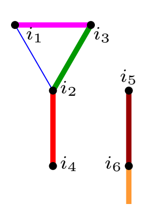

Figure 7: The part of H corresponding to this part of G'

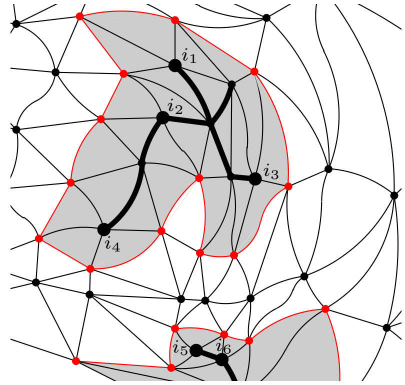

Figure 8: The corresponding part of H' denoted by bold, the U_i's denoted by grey and the c_i's denoted by red.

Lemma 15 *There exists a point $O_i \in S$ for which all the vertices of H_i' fit into a disk D_i of radius $3n_i - 3$ around O_i, all the vertices belonging to U_i fit into a disk D_i' of radius $3n-1$ around O_i and all the edges belonging to U_i fit into a disk D_i'' of radius $3n - 3$ around O_i.*

The proof is is in the Appendix.

Let c_i be the cycle that borders the connected component of $S \setminus U_i$ that contains $S \setminus D_i''$. c_1 is trivially formed by vertices having graph distance 1 from H_i' meaning that it has at most $5n_i$ vertices because of Lemma 14. Also, per definition, it is inside D_i''.

Now we have three cases:

Case 1: H is connected.

Let H_1 be the only component of H. Then $l(c_1) \leq 60$ as of Lemma 14 and its vertices fit into an open unit disk D_1' of radius 35 around O_1, while its edges fit into an open unit disk D_1'' of radius 37 around O_1.

Lemma 16 *For any vertex v of $G' \cap S_1$, its graph distance from c_1 is at most 10.*

The proof is in the Appendix.

But there is a vertex of G' inside the open unit disk around the antipodal of O_1, which has distance more than $r\pi - 36 > 20$ from D_1'. And since all of this disk is outside D_1'' (the distance of O_1 and its antipodal is at least $\pi \cdot 17.9 > 56 > 37 + 1$), the vertex inside it is a vertex of S_1, so it should have graph distance at most 10 and thus, spherical distance less than 20 from all the vertices of c_1, which is a contradiction.

Case 2:

H has two connected components and both have vertex number 6 (counted with multiplicity). Call these components H_2 and H_3

Lemma 15 and the subsequent statement yield that there exists a cycle c_2 of graph length at most 30 separating H_2' from all the vertices outside a disk D_2'' of radius 19. Similarly there exists a cycle $c_3$of graph length at most 30 separating H_3' from all the vertices outside a disk D_3'' of radius 19. Now define the interior of c_2 ($int(c_2)$) as the component of $S \setminus c_2$ containing H_2' and the interior of c_3 ($int(c_3)$) as the component of $S \setminus c_3$ containing H_3'. Since all the vertices of c_2 and c_3 have graph distance 1 from H_2' and H_3', respectively, from Lemma 13 neither H_2' has a common vertex with c_3, neither H_3' has a common vertex with c_2. So neither of c_2 or c_3 has vertices both in the interior and the exterior (the opposite component to interior) of the other one. Also, it is not possible that their interiors are covering S completely as they both fit into an open disk of radius 19 and S cannot be covered by two disks of this size. So we have two possibilities:

The interior of one of c_2 and c_3 is completely inside the interior of the other one. In this case, we can apply the argument used in Case 1 as the one having the other one in its interior also has all the vertices from I inside it, has length at most $30 < 60$ and fits into a disk of radius $19 < 35 < 37$.

In this case, define G_2 as the graph $((ext(c_2) \cup c_2) \cap (ext(c_3) \cup c_3)) \cap G'$ (where $ext(c_2)$ and $ext(c_3)$ denote the exteriors of c_2 and c_3 (the opposite side as their interiors)).

(A third possibility would be that $ext(c_2) \subseteq int(c_3)$ and $ext(c_3) \subseteq int(c_2)$, but it is clearly impossible due to their sizes.)

Now from Lemma 9 we can find a φ function running from G_2 to the infinite triangular grid T.

Now we will use the following lemma:

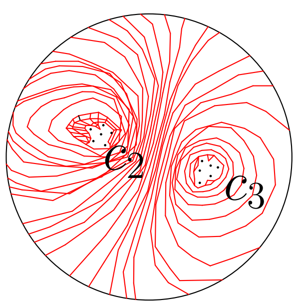

Figure 9

Lemma 17 *There exists a series of cycles ($\Gamma_0, ..., \Gamma_k$ for some k) in G' with $\Gamma_0 = c_2$ and $\Gamma_k = c_3$ satisfying 3 conditions:*

1) All of them have graph length at most 30, and thus, broken line length less than 60.

2) For any i, j with $|i - j| = 1$ and any vertex v of Γ_i, there is a vertex of Γ_j neighbouring v in G' and thus, having spherical distance less than 2 from it.

3) For any i, j with $|i - j| = 1$ and any edge e of Γ_i, there is a vertex of Γ_j having graph distance at most 1 from both of the endpoints of e in G', and thus, having spherical distance less than 4 from all of the points of e.

The proof is in the Appendix.

Now we can finish the proof for Case 2 using the following lemma:

Lemma 18 *At least one of $\Gamma_0, ..., \Gamma_k$ goes through two points that have spherical distance at least $r\pi - 5$.*

The proof is in the Appendix.

And since all of these curves have graph length at most 30, from Lemma 7 they also have broken line length less than 60. But from Lemma 18, there is one with broken line length more than $2 \cdot (r\pi - 5) \geq 2 \cdot (17.9\pi - 5) > 102$, which is a contradiction.

Case 3: Neither of the conditions of the previous cases hold.

Lemma 19 *If Case 3 holds, then there exists a cycle c_4 in G', all of whose vertices have graph distance at least 2 from all the irregular vertices and which separates them into two groups so that both of the groups has a cardinality not divisible by 6 (counted with multiplicity).*

The proof is in the Appendix.

Lemma 20 *For all vertices of c_4, a spanning subgraph of the 2-neighbourhood can be obtained as the image of an incidence and orientation preserving function Ψ from the hexagonal graph in Figure 4.*

The proof is in the Appendix.

Lemma 21 *The curvature of c_4 is divisible by 6.*

The proof is in the Appendix.

And this gives us a contradiction as the curvature of c_4 is not divisible by 6 according to Lemma 12

So it is impossible to have a nice tiling of a sphere if we are using spherical distances and $r \geq 17.9$. From this, it is impossible to have a nice tiling of a sphere if we are using Euclidean distances and $r \geq 18$ as if we have a sphere of radius $r \geq 18$ with a nice tiling, then a scaled version of the tiling would give a nice tiling for a sphere of radius $\frac{1}{2\arcsin\frac{1}{2r}} \geq 17.9$ using the spherical distance version, which is a contradiction. $\square$

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3.1 Appendix

Proof of Lemma 2

The statement not being true would mean that there is a nice tiling for some surface, for which two vertices corresponding to tiles of the same colour have graph distance 1 or 2, but in the first case, their distance on the surface would be 0, while in the second, it would be less than 1, as the diameter of their common neighbour is less than 1. So the colouring would not be nice, so the statement is true.

Proof of Lemma 3

If we take an open disk D of radius 1 around any point of A, it covers A, so $S \setminus D \subseteq S \setminus A$ and since $S \setminus D$ is connected, all of its points are in the same connected component of $S \setminus A$. And this component is unique as $S \setminus D$ is a closed disk of radius $r\pi - 1$, so none of the other components satisfy the property described in the statement of the lemma.

Proof of Lemma 4

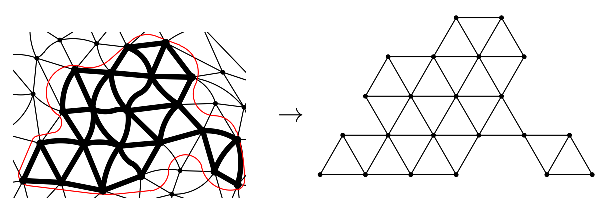
If we take an open disk D of radius 1 around a common border point of A and B, it covers both A and B, so $S \setminus D \subseteq S \setminus (A \cup B)$ and since $S \setminus D$ is connected, all of its points are in the same connected component of $S \setminus (A \cup B)$. And this component is unique as $S \setminus D$ is a closed disk of radius $r\pi - 1 > 1$, so none of the other components satisfy the property described in the statement of the lemma.

Proof of Lemma 6

If the center O of D belongs to a tile that is represented in G', then the vertex representing it has distance less than 1 from it, so it is inside D.

If O belongs to a tile that is not represented in G' because it is in the small component of $S \setminus t$ for some tile t that is represented in G' or in the small component of $S \setminus (t_1 \cup t_2)$ for some tiles t_1 and t_2 represented in G' (it is possible that O is also in the small component for some tile or tiles that are not represented in G'), then we can find a segment going through O that has both of its endpoints in the same tile represented in G' or at least on its borders and which has length less than 1. So it has distance less than 1 from the vertex representing this tile, which is thus in D.

Proof of Lemma 8



The statement can be proven by induction for the number of vertices of G^*:

If G^* does not have any vertices, the statement is trivial. Now suppose that it has m vertices and for all smaller vertex numbers, we have already proven the statement.

Suppose G^* is connected, otherwise the statement is trivial (its connected components can be defined by connected subsets of S, from which we can apply the induction hypothesis).

There is at least one (in fact, at least 4) irregular vertex in G', and it is not contained in G^*, thus if G^* is not empty, there is a vertex v inside G^*, which has at least one neighbour in $G' \setminus G^*$.

If v is a cut vertex in G^*, then examine the parts it separates G^* to (v included). All such graphs also can be determined by a simply connected subset of S and they all have a smaller number of vertices than G^* had, and since we already have φ's for these smaller graphs, by choosing $\varphi(G^*)$ arbitrarily and rotating the images of the components around it appropriately, we get an appropriate φ.

If v is not a cut vertex, then suppose there are two edges e and e' of $G' \setminus G^*$ starting from v. Then we can find points $P \in e \setminus S^*$ and $P' \in e' \setminus S^*$ that means that the curve starting from P and ending in P' going through v on e and e' separates S^* into at least two separate parts. Thus, since v is not a cut vertex, at most one of the above parts contains any vertices of G^* meaning that the edges inside G^* that are starting in v form a connected interval among all the edges in G' starting from v. And because G' being triangulated, they form a chain in which all the neighbouring pairs of edges have 1 edge between them around their common vertex in the appropriate orientation. So the φ function belonging to $G^* \setminus v$ translates them to the subset of a regular hexagon around one vertex in T, so $\varphi(v)$ can be placed in its center, and the new φ we get (by also translating the edges from v into appropriate places) satisfies the conditions.

If G^* is connected, then φ is unique with respect to isometry as starting from a vertex and deciding which direction will be which, we always can continue in only one way.

Proof of Lemma 9

Take a simply connected covering space of S'.

Again, we can suppose that we only care about connected subgraphs.

We can find a function φ' from the pre-image of G^* (call this pre-image $G^{*\prime}$) to T in the same way we did it above, despite $G^{*\prime}$ containing an infinite number of vertices and edges: we simply use induction by always expanding $G^{*\prime}$, but never deleting anything from it and also we always keep the graph connected. And since the φ' of these

And from this we get the φ function mentioned in the statement: in any vertex or edge it will take the set of the φ''s of the pre-images of the particular vertex or edge.

And this φ will satisfy the conditions of the lemma as a consequence of the definition of covering spaces.

Proof of Lemma 10

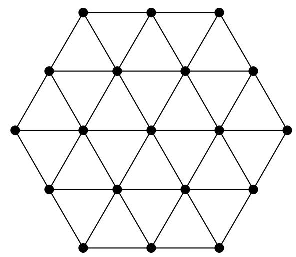


Figure 10

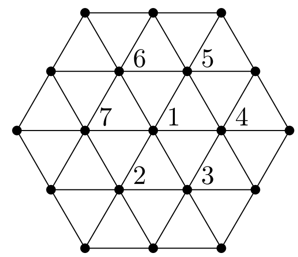


Figure 11

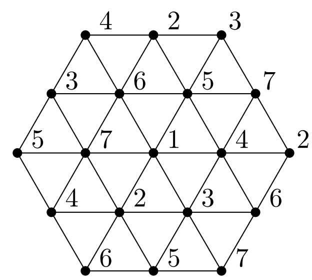


Figure 12

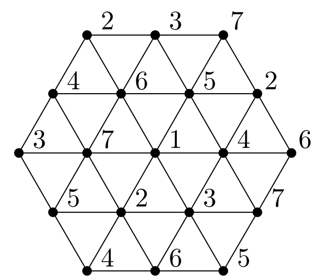


Figure 13

Colour the central vertex and its neighbours first (Figure 11). We now have to colour the remaining 12 vertices so that all border vertices of the central hexagon get exactly one neighbour from all colours (except for its own colour). And since for all colours from 2 to 7, there are exactly 3 coloured vertices that lack a neighbour with that colour and all of the uncoloured vertices border 1 or 2 of the coloured ones, we must use all six of these colours at least twice. But since there are 12 uncoloured vertices in total, we must use all of them exactly twice. From the above conditions, there are only two possibilities for choosing the vertices with colour 2 and from here, all the other colours follow (see Figure 12 and Figure 13).

Proof of Lemma 12

We will contract c into one triangle from the triangulation in the inside of c using the following steps:

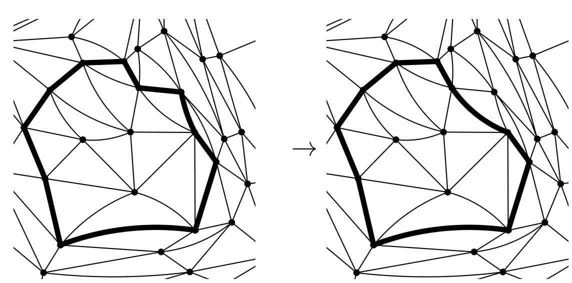


Figure 14: Step type 1

1. If we find an edge in E_i that connects two vertices v_a and v_b which are both on c and have distance 2 in

c, then we throw out the two edges between v_a and v_b and replace them with this edge and we also reduce E_i accordingly (as seen in Figure 14).

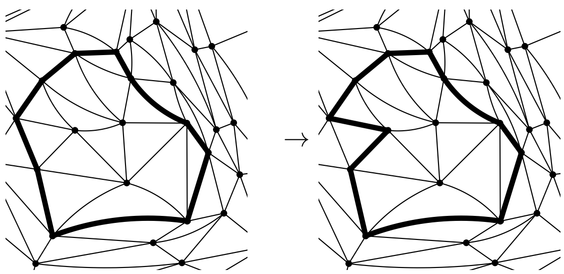


Figure 15: Step type 2

2. If we find a vertex v_c in V_i that is neighbouring two vertices v_d and v_e in c that are neighbouring each other in c, then we delete the edge v_dv_e from c and replace it with the edges v_dv_c and v_cv_e and then reduce V_i and E_i accordingly (as seen in Figure 15).

First I will prove that with always performing one of the steps above until it becomes impossible, the procedure we get is finite and ends in c being a triangle with its empty side being the inside:

If we are only looking at the subgraph of $c \cup E_i$ which is spanned by the vertices of c, it contains at most $n-3$ diagonals, so there is a vertex v_f (actually, at least 3) of c with no diagonals starting from it. And then if c is not already a triangle, the edge v_f forms with one of its neighbours (v_g) either belongs to a triangle that connects v_g with the other neighbour of v_f or to a triangle that connects both v_f and v_g with a vertex from V_i. In the first case, we can perform step 1, while in the second case, we can perform step 2. (And if c is a triangle with its empty side being the inside, then we cannot perform any of the steps.) And the number of triangles inside c always decreases with exactly 1, so the procedure is always finite.

If we perform step 1, the curvature of c does not change as the two summands belonging to v_a and v_b increase by 1, while the summand $2-0=2$ belonging to the vertex we have thrown out gets out from the sum.

If we perform step 2 and v_c is a regular vertex, the curvature of c does not change either, since the two summands belonging to v_d and v_e increase by 1 each and the new summand belonging to v_c is $2-4=-2$.

If we perform step 2 and v_c is an irregular vertex with multiplicity k, the curvature of c increases with k, since the two summands belonging to v_d and v_e increase by 1 each and the new summand belonging to v_c is $2-4+k=k-2$.

So since in the beginning all irregular vertices were in V_i and in the end no vertices remained in V_i and the curvature increased with k exactly if we deleted an irregular vertex of multiplicity k from V_i, otherwise it remained unchanged, the curvature have increased with the number of irregular vertices originally in V_i (counted with multiplicity). Also, the curvature of the triangle (with the empty side being the inside) is trivially 6, and this finishes the proof.

Proof of Lemma 13

Per definition, $dist(v, V(H_i)) \leq 1$ for all $v \in V(H_i')$. So if two vertices of some distinct graphs H_i' and H_j' have distance at most 1, then there also exist two vertices of H_i and H_j with distance at most 3, which contradicts to H_i and H_j being separate components of H.

Proof of Lemma 14

Since H_i has n_i vertices counted with multiplicity, it has at most n_i vertices counted without multiplicity. And we have drawn $|V(H_i)|-1 \leq n_i - 1$ paths all of length at most 3 (and thus all having at most 2 interior vertices) between them, thus, their union has at most $3n_i - 2$ vertices. And since H_i' is a subgraph of this union, it also has at most $3n_i - 2$ vertices. And since it is a tree, $|E(H_i')| = |V(H_i')| - 1 \leq 3n_i - 3$.

So only the third statement remains to be proved. H_i' contains n_i irregular vertices (counted with multiplicity), so $\sum_{v \in H_i'} deg_{G'}(v) = 6 \cdot |V(H_i')| - n_i$. And from these, $|\{(v,e) | v \in V(H_i'), e \in E(G' \setminus H_i'), v \in e\}| = \sum_{v \in H_i'} deg_{G'}(v) - 2 \cdot |E(H_i')| = 4 \cdot |V(H_i')| - n_i + 2$. And all edges of H_i' border exactly two triangles and all triangles are bordered by at most two edges of H_i' since H_i' is a tree. Denote the number of triangles with one side in H_i' by k_1 and the number of triangles with two sides in H_i' by k_2. From the above, we know that $k_1 + 2k_2 = 2 \cdot |E(H_i')| = 2 \cdot |V(H_i')| - 2$. The number of edges of $G' \setminus H_i'$ connecting two vertices of H_i' is at least k_2, since any triangle with two sides in H_i' has such an edge as its third edge, and such an edge cannot belong to two different such triangles as then there would be a 4-cycle in H_i'. Thus, the number of edges of G' having one endpoint in H_i', while the other one in $G' \setminus H_i'$ is at most $|\{(v,e) | v \in V(H_i'), e \in E(G' \setminus H_i'), v \in e\}| - 2k_2 = 4 \cdot |V(H_i')| - n_i + 2 - 2k_2$ as all such edges are in $G' \setminus H_i'$ and all edges of $G' \setminus H_i'$ connecting two vertices from H_i' were counted twice in the above calculation. Now take a vertex v for which $dist_{G'}(v, H_i') = 1$ and take the set τ of triangles which have a side in H_i' and their third vertex is v. It is trivial that every such triangle is bordered by exactly two edges connecting v to H_i' and all edges are contained as an edge of at most two such triangles. Also, the latter inequality cannot be strict in case this particular edge is also bordering a triangle that is not in τ. So at least one of the following possibilities hold:

1) There are more than $|\tau|$ (so at least $|\tau|+1$) edges connecting v with H_i'

2) No edges connecting v with H_i' border the union of the triangles from τ.

But 2) is only possible if τ is empty (in which case,

1) still holds as there is at least one edge connecting v with H'_i) or if τ contains all triangles bordering v in which case the cycle formed by the neighbours of v is a subgraph of H'_i, which is a contradiction as H'_i is a tree. So 1) holds.

And by summing up such inequalities for all v's having graph distance 1 from H'_i, we get that $|\{v|v \in V(G'), dist_{G'}(v, H'_i) = 1\}| \leq |\{(v,w)|(v,w) \in E(G'), v \in V(H'_i), w \in V(G \setminus H'_i)\}| - k_1 \leq 4 \cdot |V(H'_i)| - n_i + 2 - 2k_2 - k_1 = 2 \cdot |V(H'_i)| + 4 - n_i \leq 5n_i$.

Proof of Lemma 15

Since $|V(H'_i)| \leq 3n_i - 2$, if H' is a centered tree, there is a vertex with graph distance at most $1.5n_i - 1$ from all of its vertices, while if it is a bicentered tree, there is an edge, whose endpoints both have graph distance at most 16 from all of its vertices. In the former case, the vertices of H' fit into an open disk of radius 32 because of Lemma 7, while in the latter case, the vertices of H' fit into an open disk of radius 33 because of Lemma 7. So in both cases the vertices of H' fit into a disk D_1 of radius 33 centered around a point on S called O_1. And since the vertices of U have graph distance at most 1 from H'_i, they

Proof of Lemma 16

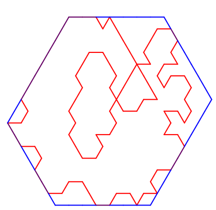


Figure 16

Take the cycle c'_1 in T represented by the blue cycle in Figure 16, where the red one represents $\varphi(c_1)$ (and the purple segments are their common edges). If we name the length of the ith side of c'_1 A_i for $1 \leq i \leq 6$ and we choose a vertex p_i of $\varphi(c_1)$ on all 6 of them, then for all i ($i = 1, 2, 3$) the two parts of $\varphi(c_1)$ separated by p_i and p_{i+3} both have at least $|A_{i+1}| + |A_{i+2}|$ segments parallel with A_{i+1} or A_{i+2} (i counted modulo 3), so $\varphi(c_1)$ has at least $2|A_{i+1}| + 2|A_{i+2}|$ such segments in total. And from summing up the three inequalities we get this way, if we combine it with $|A_i| + |A_{i+1}| = |A_{i+3}| + |A_{i+4}|$, we get $l(c_1) = l(\varphi(c_1)) \geq \sum_{i=1}^{6} |A_i| = l(c'_1)$ (the first equality is trivial).

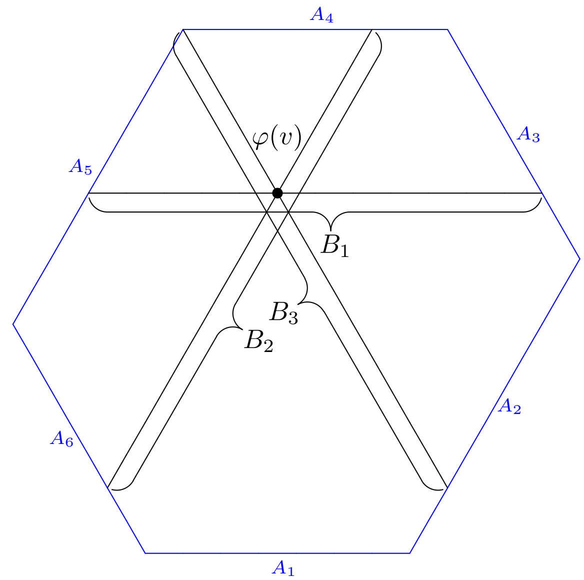


Figure 17

Now take a vertex v in G_1. If we call the segments going through $\varphi(v)$ parallel with A_1, A_2 and A_3 and ending in c'_1 B_1, B_2 and B_3, respectively (as in Figure 17), then we can write the following inequalities:

$|B_i| \leq |A_i| + \frac{|A_{i-1}|}{2} + \frac{|A_{i+1}|}{2}$ (for $i = 1, 2, 3$ if i is counted $\mod 6$)

$|B_i| \leq |A_{i+3}| + \frac{|A_{i+2}|}{2} + \frac{|A_{i+4}|}{2}$ (for $i = 1, 2, 3$ if i is counted $\mod 6$)

Summing up these 6 inequalities, we get $\sum_{i=1}^{3} 2|B_i| \leq \sum_{i=1}^{6} 2|A_i|$ and if we combine this with the fact that the distance of $\varphi(v)$ from c'_1 is at most $\frac{\min(|B_1|,|B_2|,|B_3|)}{2}$ we get $dist_{G'}(\varphi(v), c'_1) \leq \left\lfloor \frac{l(c'_1)}{6} \right\rfloor \leq \left\lfloor \frac{l(c_1)}{6} \right\rfloor \leq 10$ (where $dist_{G'}(\varphi(v), c'_1)$ denotes the graph distance of $\varphi(v)$ and c'_1 in G'.

Now if we define P_{min} as a path of minimal length from $\varphi(v)$ to c'_1, we can start a path from v so that the images of its vertices by φ are the vertices of P_{min} in the same order. And we can always take a step such that the image of the next vertex will be the next vertex in P_{min} until we reach c_1. And this can happen the latest when the φ of the path reaches c'_1, so $dist_{G'}(v, c_1) \leq dist_{G'}(\varphi(v), c'_1) \leq 10$.

Proof of Lemma 17

We will first contract G_2 into a cycle using the following six kinds of steps:

(The figures below represent the $\varphi(G_2)$ and $\varphi(c_2)$, which means that red broken line ($\varphi(c_2)$) occasionally can seemingly cross itself. From the curvature of both c_2 and c_3 being 0, it is easy to see that the sum of the angles of the turns is 0 for both $\varphi(c_2)$ and $\varphi(c_3)$, thus for any vertex or edge from G_2, its images by φ are periodical translates of each other.)

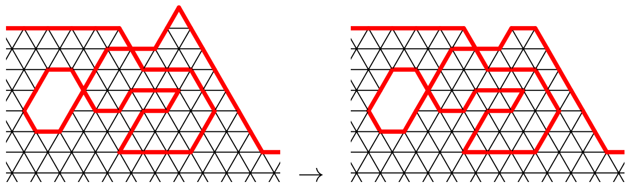


Figure 18: A vertex with degree 2 in G_2 is removed from G_2 and c_2 is changed accordingly.

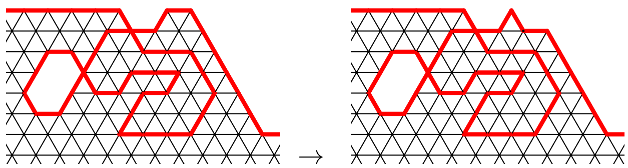


Figure 19: A vertex with degree 3 in G_2 is removed from G_2 and c_2 is changed accordingly.

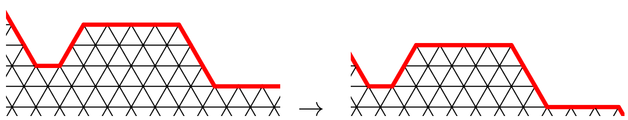


Figure 20: c_2 is shifted in some direction towards c_3 without adding anything to G_2.

The other three possibilities are doing the same kinds of steps with c_3.

Now we will prove that with such steps, we always can contract G_2 into a cycle without adding any vertex to G_2 in any step.

First, suppose that c_2 and c_3 have no common edges. Then if any of $\varphi(c_2)$ and $\varphi(c_3)$ is not a line, we can find a convex turn in one of them (a vertex of c_2 or c_3 in which $deg_{G_2}(c_2) \leq 3$ or $deg_{G_2}(c_3) \leq 3$, respectively) because both of them has a curvature of 6. So one of the first two kind of steps (or their c_3 counterparts) can be applied.

Now suppose that c_2 and c_3 have no common edges, but both of them are a line. In this case, we can apply Step type 3 without adding any vertex to G_2.

Now suppose that there is at least one edge in which c_2 and c_3 meet. In this case there is at least one subset $\hat{S}$ of S between c_2 and c_3 that is simply connected and is fully bordered by c_2 and c_3. And if we count $\hat{S}$ as the interior, its border has a curvature of 6, so even if in the two vertices, where c_2 and c_3 meet, the border takes a sharp convex turn (one, in which the degree towards $\hat{S}$ is 2, it still has to take a convex turn somewhere elsewhere, in which case, we can leave this particular vertex from c_2 or c_3.

Thus, the only case we cannot take such a step is if c_2 and c_3 coincide. And since the number of triangles between c_2 and c_3 always decreases with at least one, in finitely many steps, the procedure ends in G_2 being a cycle.

Now take every step in which c_2 was changed and name the original c_2 $c_2^{(0)}$, c_2 after the first step changing it $c_2^{(1)}$, c_2 after the second such step $c_2^{(2)}$, ... until we get to $c_2^{(p)}$ (where p is the number of steps changing c_2). Similarly, if the number of steps changing c_3 is q, we can define cycles $c_3^{(0)}, c_3^{(1)}, ..., c_3^{(q)}$.

Now take $k = p + q$ and define $\Gamma_0 = c_2^{(0)}$, $\Gamma_1 = c_2^{(1)}$, ..., $\Gamma_p = c_2^{(p)} = c_3^{(q)}$, $\Gamma_{p+1} = c_3^{(q-1)}$, ..., $\Gamma_k = p_3^{(0)}$.

For these cycles, the first condition of the lemma trivially applies, since all of the above steps decrease the graph length for both c_2 and c_3, so since originally, c_2 and c_3 did not have graph length more than 30, none of the Γ_i's ($i = 0, 1, ..., k$) have. So their broken line length is not more than 60.

The second and the third condition also can easily checked for both the $c_2^{(i)}$ ($i = 0, 1, ..., p$) and the $c_3^{(i)}$ ($i = 0, 1, ..., q$)

Proof of Lemma 18

First, for all $i = 0, ..., k$ let Γ_i' be the antipodal curve of Γ_i. Now define $int(\Gamma_i)$ as the connected component of $S \setminus \Gamma_i$, which includes the vertices of I_2, and let the other connected component be called $ext(\Gamma_i)$. Now define $int(\Gamma_i')$ and $ext(\Gamma_i')$ as the antipodal sets of $int(\Gamma_i)$ and $ext(\Gamma_i)$, respectively. Now define a function $f(i) = dist_S(V(\Gamma_i), ext(\Gamma_i')) - dist_S(V(\Gamma_i), int(\Gamma_i'))$. The first half is 0 if and only if at least one of the vertices of Γ_i is in $ext(\Gamma_i') \cup (\Gamma_i')$, while the second half is 0 if and only if at least one of the vertices of Γ_i is in $int(\Gamma_i') \cup (\Gamma_i')$. Thus, at least one of the two halfs is always zero and the sign of $f(i)$ is determined by which one is not zero. And since $V(\Gamma_0) \subseteq D_2''$ and $int(\Gamma_0')$ is inside the antipodal of D_2'', they are disjoint and their distance is positive. Similarly, $V(\Gamma_k)$ and $ext(\Gamma_k)$ are disjoint and their distance is positive. Thus, $f(0) < 0$ and $f(k) > 0$. Now take the first i, for which $f(i)$ is positive. If $f(i-1) \leq -3$, that means that all the vertices of Γ_{i-1} have a distance at least 3 from $int(\Gamma_{i-1}')$, so the vertices of Γ_i are further from $int(\Gamma_{i-1}')$ than 1, since all of them has distance less than 2 from at least one of the vertices of Γ_{i-1}. But also, for all border points of $int(\Gamma_{i-1})$, there exists a point of $ext(\Gamma_i)$ with distance less than 4 from it, so $f(i) = dist_S(V(\Gamma_i), ext(\Gamma_i)) < 4 - 1 = 3$. Thus, at least $|f(i-1)| < 3$ or $|f(i)| < 3$ is true, so one of the two cycles includes a vertex and another point, whose antipodals have spherical distance 3. And since the latter is 2 away from a vertex of the same cycle, we have found two vertices on one of the cycles with distance at least $r\pi - 5$ from each other.

Proof of Lemma 19

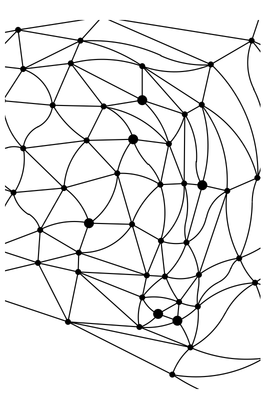


Figure 21: A part of G

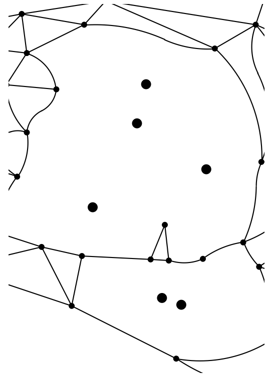


Figure 22: The same part of G_3

Let the components of H be H_4, H_5, ... Now take the graph G_3 which we get from G' by deleting all the vertices that have graph distance at most 1 (with respect to G') from any of the points of I and all the edges that are incident to these vertices. All the irregular vertices, which are in different connected components of H are in different connected components of $S \setminus G_3$, since if there would be a path on S connecting two irregular vertices in different components of H, the triangles, edges and vertices it meets would all have graph distance at most 1 from some irregular vertex, which gives us a contradiction.

Now take an H_a which does not have a vertex number divisible by 6 (counted by multiplicity). If the point set G_3 contains a simple closed curve that separates the vertices of H_a from all the other irregular vertices, then this curve only can be a cycle in G_3 and it is applicable for c_4. If G_3 does not contain such a curve, it only can mean that the connected component of $S \setminus G_3$ belonging to I_i (call it C_i) separates the rest of S into at least two parts of which at least two contains a positive number of irregular vertices. Then we can separate C_i from both of these two parts by a cycle c_a and c_b, and it is trivial that both of these cycles separate the irregular vertices into two non-empty sets and at least one of the cycles divides I unevenly, so it can be chosen for c_4.

Proof of Lemma 20

Call the vertices of c_4 $u_0, ..., u_{l(c_4)-1}$ in the positive order they appear in c_4.

Take an arbitrary vertex $u_i \in c_4$. The neighbours of u_i form a 6-cycle (call its vertices $w_{i,0}, ..., w_{i,5}$ in the positive order they appear in the cycle) because they are connected by the sides opposite to u_i of the triangles having u_i as a vertex. Since not only u_i, but also its neighbours are regular vertices, they all have exactly 3 edges remaining and for any $w_{i,j}$, these remaining vertices are forming an interval in $n_1(w_{i,j})$. And from the triangulatedness of G', for any j, the rightmost of these three neighbours of $w_{i,j}$ is the same as the leftmost one of $w_{i,j-1}$ (counted $\mod 6$), while the central one is connected with the other two. So if we name the common neighbour of $w_{i,j-1}$ and $w_{i,j}$ as $t_{i,j-1,j}$ and the common neighbour of $w_{i,j}$, $t_{i,j-1,j}$ and $t_{i,j,j+1}$ as $t_{i,j}$, then all the drawn edges exist. So although the $N_2(u_i)$ might not be isomorphic with the graph in Figure 4 because of the $t_{i,j}$'s and $t_{i,j-1,j}$'s not being regular (it is possible, that some of the vertices listed above coincide or there exist edges not shown in the drawing), a (spanning) subgraph of it can be obtained as the result of a Ψ_i function preserving incidence between vertices and edges and is also preserving triangles and orientation. Also, once we have decided the rotation of Ψ_i regarding the neighbours of $\Psi^{-1}(u_i)$, the function is unique.

Proof of Lemma 21

For all i ($i = 0, ..., l(c_4) - 1$) let the Isbell colouring we coloured $\Psi_i^{-1}(N_2(u_i))$ with be named χ_i.

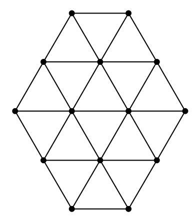


Figure 23

Similarly as in Lemma 10, there is a unique function preserving incidences between vertices and edges that projects the graph in Figure 23 into $N_2(u_i) \cap N_2(u_{i+1})$ counted $\mod l(c_4)$ for $i = 0, ..., l(c_4) - 1$. Thus, we can suppose Ψ_i and Ψ_{i+1} are the same on $N_2(u_i) \cap N_2(u_{i+1})$. And since this graph contains a subgraph that is isomorphic with that in Figure 5, from Lemma 11 we know that $\chi_i = \chi_{i+1}$. Thus, $\Psi_i^{-1}(N_2(u_i))$ is coloured with the same Isbell colouring for all $i = 0, ..., l(c_4) - 1$. And since Ψ_i is an isomorphism on $N_1(u_i)$, the $N_1(u_i)$ are also all coloured with the same Isbell colouring.

Now colour T with this colouring (call it χ_0) and define a function g from ordered pairs of colours to directions in T: for any ordered pair of colours, take the ordered pairs of neighbouring vertices of T coloured with these two colours. It is trivial from the definition of an Isbell colouring that the direction of the vector connecting the two members of such an ordered pair is uniquely defined by the ordered pair of the colours. Let this direction be the g of this pair. Also define a function h from the ordered pairs of vertices in c_4, which is defined as the g of the ordered pair of colours belonging to the particular vertices. And from $N_1(u_1)$ being coloured with χ_0, we know that $\angle h(u_{i-1}, u_i), h(u_i, u_{i+1}) = (2 - (e(u_i, int(c_4)))) \cdot \frac{\pi}{3}$, where $int(c_4)$ is defined as the side it goes around in a positive direction). And from this and $\sum_{i=0}^{l(c_4)-1} \angle(f(u_i, u_{i+1}), f(u_{i+1}, u_{i+2})) = 0$, we get that the curvature of c_4 is divisible by 6.